Exercise 2

A spring with an 8-kg mass is kept stretched 0.4 m beyond its natural length by a force of 32 N. The spring starts at its equilibrium position and is given an initial velocity of 1 m/s. Find the position of the mass at any time t.

Solution

In order to determine the spring constant, use the fact that 32 N is needed to stretch the spring 0.4 m.

$$F = k(x - x_0)$$

$$32 \text{ N} = k(0.4 \text{ m})$$

$$k = 80 \ \frac{\text{N}}{\text{m}}$$

Apply Newton's second law to obtain the equation of motion.

$$F = ma$$

Use the fact that acceleration is the second derivative of position $a = d^2x/dt^2$ and the fact that the spring force $F = -kx$ is the only force acting on the mass.

$$-kx = m\frac{d^2x}{dt^2}$$

Divide both sides by m.

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x = e^{rt} \quad \rightarrow \quad \frac{dx}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$r^2e^{rt} = -\frac{k}{m}e^{rt}$$

Divide both sides by e^{rt}.

$$r^2 = -\frac{k}{m}$$

Solve for r.

$$r = \left\{-i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}}\right\}$$

Two solutions to the ODE are $e^{-i\sqrt{k/m}t}$ and $e^{i\sqrt{k/m}t}$.

By the principle of superposition, then, the general solution to equation (1) is

$$\begin{aligned}
x(t) &= C_1 e^{-i\sqrt{k/m}t} + C_2 e^{i\sqrt{k/m}t} \\
&= C_1\left(\cos\sqrt{\frac{k}{m}}t - i\sin\sqrt{\frac{k}{m}}t\right) + C_2\left(\cos\sqrt{\frac{k}{m}}t + i\sin\sqrt{\frac{k}{m}}t\right) \\
&= (C_1 + C_2)\cos\sqrt{\frac{k}{m}}t + (-iC_1 + iC_2)\sin\sqrt{\frac{k}{m}}t \\
&= C_3\cos\sqrt{\frac{k}{m}}t + C_4\sin\sqrt{\frac{k}{m}}t,
\end{aligned}$$

where C_3 and C_4 are arbitrary constants. Differentiate it with respect to t to get the velocity.

$$\frac{dx}{dt} = -C_3\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + C_4\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t$$

Apply the initial conditions, $x(0) = x_0 - x_0 = 0$ and $x'(0) = 1$, to determine C_3 and C_4.

$$x(0) = C_3 = 0$$

$$\frac{dx}{dt}(0) = C_4\sqrt{\frac{k}{m}} = 1$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = \sqrt{m/k}$.

$$x(t) = \sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t$$

Therefore, plugging in $m = 8$ kg and $k = 80$ N/m, the displacement from equilibrium is

$$x(t) = \frac{1}{\sqrt{10}}\sin\sqrt{10}t.$$

Below is a plot of $x(t)$ versus t.